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Minimum Energy Requirements of Information Transfer and Computing

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The minimum energy requirements of information transfer and computing are estimated from the time-energy uncertainty relation.

1. INTRODUCTION

In 1961/1962 this author proposed that the uncertainty principle of quantum mechanics imposes fundamental limitations upon computers (Bremermann, 1962a, 1962b). As Von Neumann (1958) pointed out, computers represent data by physical markers. In the course of computation, data are transferred between subunits. This implies that the physical markers that represent data must be measured whenever data are stored, retrieved, transmitted, or entered into the arithmetic unit, central unit, etc.

Measurements are subject to error margins because of the Heisenberg uncertainty principle of quantum mechanics. In practice signal energies are large and markers are spaced far enough apart for quantum uncertainties to be insignificant. As computers are being miniaturized and signal energies are reduced the safety margins are shrinking, while at the same time computing power per pound of hardware is increased. Eventually a limit is approached where the quantum uncertainties prevent further miniaturization and an upper limit of computing power is reached.

Data processing in a Von Neumann-type computer consists of several kinds of operations: input, output, storage and retrieval, logical and arithmetic operations, and control. Transmission between subunits lends itself most easily to a quantum uncertainty analysis. A unit that receives data from another unit must interpret the markers that are transmitted. We propose that the receiving unit is subject to the same uncertainty laws as the entity of quantum mechanics known as *observer*. In other words, the receiver (input) measures physical observables and it makes no difference, for the purpose of attainable accuracy, whether the observables affect a modular subunit of a computer or a human observer who is coupled to the measurement apparatus. If in a computer subunits are not uncoupled in this way they should not be considered as separate but as a single system.

In both his 1962a and 1967a papers this author proposed to apply Shannon's theory of the transmission capacity over a noisy, continuous channel to the problem. Shannon's formula for the capacity C of a noisy channel is

$$C = \text{bandwidth} \times \log(1 + S/N)$$

where *bandwidth* is the width of the frequency band transmitted (in cycles/sec), and S/N is the *signal-to-noise power ratio.* C is the maximum rate (in bits per second) at which data can be transmitted and recovered (Shannon, 1948).

The bandwidth open in a quantum channel is limited by $\nu_{\max} = E_{\max}/h \leqslant mc^2/h$, where $E_{\max}$ is the maximum energy available, divided by Planck's quantum since for $\nu > \nu_{\max}$ there is not enough energy to produce a single photon. We also assume that photons would be the most efficient use of mass/energy for signaling since they are the only particles without a rest mass. The author proposed that energy–time uncertainty relation could be interpreted as generating a *quantum noise* such that the signal-to-noise power ratio equals 1 (This ratio has now been revised to 4π).

This results in an upper limit of information transmission of $C \leqslant (mc^2/h)\ln(1+4\pi)$.

The author's argument about the signal-to-noise ratio has been questioned by Bekenstein (1981), who recently rederived the author's result in a framework of cosmology, general relativity, and the thermodynamics of black holes. Bekenstein's method is based on thermodynamics as generalized to a relativistic system involving black holes that radiate à la Hawking. In the work of Bekenstein (1973, 1980, 1981a,b) and Hawking (1975, 1975) the entropy of a system with a finite effective radius has a bound which implies that the amount of information (number of bits) that can be encoded in the distinguishable states of the system is bounded. From this bound Bekenstein derives a bound for the amount of information that is transmissible per unit of mass and energy per second. This bound is the same as the author's except for a numerical factor of $2\pi^2/\ln 2$ versus $\ln(1+4\pi)$.

The author's argument in contrast is based on the time–energy uncertainty principle of quantum mechanics and does not involve general relativity, curved space, singularities, or cosmology. In the following we will show that the uncertainty relation for time and energy does indeed give a signal-to-noise ratio $S/N = 4\pi$ and that consequently, our result follows from Shannon's formula. The author's result thus seems to form a logical link between the entropy bound in a relativistic universe including black holes and the uncertainty principle.

In view of the fundamental nature of the result and because of the extraordinary problems that have arisen in a precise interpretation of the time–energy uncertainty relationship, we will discuss it in the following in some detail. Basically it is a mathematical phenomenon in Fourier theory. In the spirit of this conference we "squeeze quantum effects into a small corner."

2. UNCERTAINTY RELATIONS AND THEIR INTERPRETATION

The uncertainty principle, classically, applies to the simultaneous measurement of position and momentum of a particle in nonrelativistic quantum mechanics. Let ψ be the time-dependent wave function describing the particle. Let x be the position operator (which describes measurement of position) and p the momentum operator; then

$$(\Delta x)^2 = \frac{\langle\psi|(x - x_0)|\psi\rangle}{\langle\psi|\psi\rangle}$$

and

$$(\Delta p)^2 = \frac{\langle\psi|(p - p_0)^2|\psi\rangle}{\langle\psi|\psi\rangle}$$

where

$$x_0 = \frac{\langle\psi|x|\psi\rangle}{\langle\psi|\psi\rangle}$$

$$p_0 = \frac{\langle\psi|p|\psi\rangle}{\langle\psi|\psi\rangle}$$

Here x_0 and p_0 are the expectation values of the measurements and Δx and Δp the second moments (variances) of the probability distributions of measured values. The uncertainty relation states that

$$\Delta x \, \Delta p \geqslant \frac{\hbar}{2} = \frac{h}{4\pi}$$

where h is Planck's quantum. (We note that different authors write $\hbar$ as well as h instead of $\hbar/2$.)

Position and momentum are conjugate coordinates, their product has the dimension of *action* = *energy* $\times$ *time*. In special relativity time and space variables are connected by Lorentz invariance, as are energy and momentum. Thus one expects for reasons of symmetry the analogous energy–time uncertainty relation $\Delta E \Delta t \geqslant \hbar/2$.

The interpretation of this energy–time uncertainty relation, however, poses special problems (Allcock, 1969; Wigner, 1972). Allcock's papers contain a review of the literature of the time–energy uncertainty principle as well as criticism of imprecise interpretations. Allcock expresses pessimism about the possibility of incorporating it into the present framework of quantum mechanics. Wigner (1972) points out that it is best not to interpret it *in abstract* but for specific instances such as the lifetime versus energy spread of excited states and unstable particles or the arrival of a particle in a fixed plane in space. The position–momentum uncertainty relation in nonrelativistic quantum mechanics refers to the position and momentum operators at an instant of time and hence to the uncertainties at an instant of time. Berestetskii et al. (1971) point out that measurements take time, and that shorter and shorter measurements would involve higher and higher velocities. Since velocities are limited by the light velocity, measurements (and uncertainties) at an instant of time are unphysical idealizations.

Wigner (1972) refers to another difficulty for the time–energy uncertainty relation in the nonrelativistic theory: In this theory there is a lower bound for the energy, while the only functions that satisfy the uncertainty equality are Gaussians (functions which do not have lower bounded support). Thus all physical wave functions satisfy at most the inequality, and in some instances a sharper inequality (Wigner, 1972). Mathematically, however, the Gaussians can be approximated arbitrarily closely by functions with compact support such that the mathematical uncertainty relation (cf. Papoulis, 1962) for the function and its Fourier transform is approximated arbitrarily closely. Hence in general, the uncertainty relation is sharp, even in nonrelativistic quantum mechanics.

The connection with physics occurs because the energy representation of the wave function is identical with the Fourier transform of its time representation, multiplied by $\hbar$.

3. THE MATHEMATICAL UNCERTAINTY RELATION

Let $f \in L_2(-\infty, \infty)$, and $tf \in L_2(-\infty, \infty)$ meaning that $f(t)$ and $tf(t)$ are square integrable over the interval $(-\infty, \infty)$. Let

$$\|f\|^2 = \int_{-\infty}^{\infty} |f(t)|^2 \, dt$$

and

$$F(\omega)=\int_{-\infty}^{\infty} f(t)e^{i\omega t}\,dt$$

be its Fourier transform. Then $2\pi\|f\|^2=\|F\|^2$. In the following we normalize $\|f\|^2=(1/2\pi)\|F\|^2=1$. Let

$$t_0=\int_{-\infty}^{\infty} t|f(t)|^2\,dt$$

$$\omega_0=\int_{-\infty}^{\infty} \omega|F(\omega)|^2\,d\omega$$

Define:

$$(\Delta t)^2=\int_{-\infty}^{\infty}(t-t_0)^2|f(t)|^2\,dt$$

$$(\Delta\omega)^2=\int_{-\infty}^{\infty}(\omega-\omega_0)^2|F(\omega)|^2\,d\omega$$

Then

$$\Delta t\,\Delta\omega \geqslant \left(\frac{\pi}{2}\right)^{1/2}$$

Equality holds if and only if

$$f(T)=\left(\frac{\alpha}{\pi}\right)^{1/2}\exp\left[-\alpha(t-t_0)^2\right], \qquad \alpha>0$$

and

$$F(\omega)=\left(\frac{\pi}{\alpha}\right)^{1/2}\exp\left(i\omega t_0-\omega^2/4\alpha\right)$$

Thus equality holds if and only if both $f(t)$ and $|F(\omega)|$ are Gaussians and the meaning of the uncertainty relation is "If one Gaussian is narrow its Fourier transform is a broadly spread Gaussian, and vice versa." [The results of this section may be found in texts on Fourier transforms such as Papoulis (1962)].

4. ENERGY–TIME UNCERTAINTY RELATION

Wigner (1972) tackled the time–energy uncertainty relation in the framework of ordinary quantum mechanics as follows: He proposed to single out a position coordinate, x (analogous to the time coordinate which is singled out in conventional nonrelativistic quantum mechanics) and writes

$$(\Delta T)^2 = \frac{\iiint |\psi(x, y, z, t)|^2 (t - t_0)^2 \, dy \, dz \, dt}{\iiint |\psi(x, y, z, t|^2 \, dy \, dz \, dt}$$

and

$$(\Delta E)^2 = \frac{\iiint |\phi(x, y, z, t)|^2 (E - E_0)^2 \, dy \, dz \, dE}{\iiint |\phi(x, y, z, t)|^2 \, dy \, dz \, dE}$$

where

$$\phi(x, y, z, E) = \int \psi(x, y, z, t) e^{iEt/\hbar} \, dt$$

He points out that the wave function can be replaced by

$$\int \rho(r)^* \psi(x, r) \, dr$$

where $r = (y, z, t)$ and ρ is an arbitrary function of r, not depending upon x. The resulting uncertainty relation then is valid for an instant of x, that is, $x = \text{const}$, which is a plane in space.

We now assume that by suitable choice of $\rho(r)$ we can describe the interaction of the wave function ψ with a measurement apparatus located in the plane $x = \text{const}$. This could be a photon detector, or electron detector, etc.

We thus have an uncertainty relation for

$$\chi(t) = \iint \rho(y, z, t) \psi(y, z, t) \, dy \, dz$$

where χ is the averaged wave function of ψ averaged with ρ. Its energy representation is

$$\eta(E) = \iint \phi(x, y, z, E) \, dy \, dz = \int \chi(t) e^{iEt/\hbar} \, dt$$

Since the energy representation equals the mathematical Fourier transform of $\chi(t)$ at $\omega = E/\hbar$: $\eta(E) = \mathfrak{F}(\chi, E/\hbar)$ then

$$\|\eta\|_E^2 = \int |\eta(E)|^2 \, dE = \hbar \int |\mathfrak{F}(\chi, \omega)|^2 \, d\omega = \hbar 2\pi \|\chi\|^2$$

Note that $(\Delta E)^2 = \hbar^3 (\Delta\omega)^2 / \|\eta\|_E^2$. Hence

$$\Delta E \Delta t = \frac{\hbar \, \Delta\omega \, \Delta t}{(2\pi)^{1/2} \|\chi\|} \geqslant \frac{\hbar}{(2\pi)^{1/2}} \left(\frac{\pi}{2}\right)^{1/2} = \frac{\hbar}{2}$$

Note: We obtain the uncertainty relations with $\hbar/2$ rather than $\hbar$. Wigner and many authors write $\hbar/2$, but other authors, for example, Berestetskii et al. (1971), write $\hbar$. These differences may arise from different definitions of the Fourier transform and the inverse Fourier transform where different authors place a factor 2π either in the forward or inverse transform or divide it between the transforms.

Note that the uncertainty relation is basically a mathematical result about Fourier transforms. All the physics is contained in the interpretation of the energy representation of the wave function and its second moment.

5. APPLICATION OF THE UNCERTAINTY RELATION TO SIGNAL TRANSMISSION

For the physical transmission of signals a *carrier* is required, e.g., photons or electrons, or possibly other kinds of particles.

In 1948 Claude Shannon analyzed the transmission of information over the continuous, band-limited, noisy channel. A channel is defined by an ensemble of functions which are admissible as *signals* $s(t)$ and which can occur as *noise* $n(t)$. It is assumed that $s(t)$ is transmitted and $s(t) + n(t)$ is received.

Let

$$S^2 = \frac{1}{\Delta T} \int_0^{\Delta T} s^2(t) \, dt = \text{signal power}$$

and

$$N^2 = \frac{1}{\Delta T} \int_0^{\Delta T} n^2(t) \, dt = \text{noise power}$$

Let W be the bandwidth of the channel.

For white noise (flat band-limited spectrum) Shannon derives for the transmission capacity of the band-limited channel the following theorem:

Let

$$C = W \log\left(1 + \frac{S}{N}\right)$$

If R is the (informational) entropy per second of a source (measured in bits per second) and if $R < C$, then it is possible to make the error rate of transmission arbitrarily small, by suitable encoding and decoding. For $R > C$ this is not possible.

Consider now a physical information channel. Let E_{max} be the maximal signal energy. Photon quanta carry an energy of $h\nu$. Thus a limitation of the signal energy limits the maximum frequency of the photons to

$$\nu_{max} = \frac{E_{max}}{h}$$

since for frequencies exceeding ν_{max} there is not enough energy for a single photon.

Suppose the signal is represented by the function $\chi(t)$ derived in the previous section. How can $\chi(t)$ be measured? We will assume that the photon channel is as efficient a use of signaling energy as any other physical channel. Photons have no rest mass and thus require no energy investment in rest mass. We assume that photons can be used as efficiently as any other particles for signaling. Particles with mass, such as electrons and heavier particles, would use up large amounts of the total energy in their mass. Moving masses can be detected by measuring their momentum, which is again subject to the uncertainty principle. Moving charges can be detected through radiation, which takes us back to photons. Or particles can be detected through interaction with other particles, but these interactions ultimately seem to be detected by photon and/or momentum measurements (cf. Feynman et al. 1963).

A full analysis is beyond our means. Quantum mechanics has conveniently buried all the problems of measurement and the entity called *observer* in the operator formalism.

We will assume that ultimately, whatever the details, measurements reduce to energy–frequency measurements, that such measurements take time, and that the accuracy depends upon the length of time available for measurement. We interpret the time–energy uncertainty relation to mean that no matter how energy is measured during an interval ΔT there remains an uncertainty

$$\Delta E \geqslant \frac{\hbar}{2\,\Delta T}$$

for a normalized wave function. The uncertainty of the energy signal applies to the entire length of the measurement interval. We interpret it as constituting white noise with a flat spectrum over the frequency band, in other words noise, with the characteristics required by Shannon's theorem. (Shannon's analysis shows that the shape of the noise spectrum has some but limited influence on C.)

The energy of the signal is given by $\|\eta\|_E^2 = \hbar 2\pi \|\chi\|^2$, where $\|\chi\|$ is normalized to 1. Hence the signal power S (energy per unit of time) is given by $\hbar 2\pi/\Delta T$. Hence the signal-to-noise power ratio

$$\frac{S}{N} = \frac{\hbar 2\pi}{\Delta T} \cdot \frac{2\,\Delta T}{\hbar} = 4\pi$$

Note that $S/N = 4\pi$ independent of the length of ΔT. Thus the fastest rate at which information can be transmitted over an energy limited channel is

$$C = \frac{E_{\max}}{h} \ln(1+4\pi) = \frac{mc^2}{h} \ln(1+4\pi) \qquad [(\text{bits/sec})]$$

Except for the factor $\ln(1+4\pi)$ this is the same figure as derived in Bremermann (1962a, 1967a).

We note that the bound on the rate of information transmission is unaffected when the total signal energy is divided between two or more channels and it is independent of the total length of transmission. If it were otherwise nonlinearities would arise and make matters very complicated.

6. ANALOG AND PARALLEL COMPUTERS

The Von Neumann-type computer (cf. Arbib, 1964) is a model of computation slanted towards *sequential*, logical and numerical computation. The human brain, in contrast, processes data in *parallel*. The intrinsic complexity of many mathematical tasks (cf. Bremermann, 1974, 1979) can make parallel computation intrinsically faster for some tasks (cf. Csanky and Bremermann, 1976; Csanky, 1976), though not necessarily for all.

The limit of the rate of information processing affects parallel computers the same as it affects Von Neumann-type computers, except that a parallel computer may contain a greater number of internal data transmission channels.

The same statement can be made for finite-state automata, except that the subunits in a finite-state automaton are themselves finite-state automata and are not restricted to arithmetic units, storage, control units, etc., as in a Von Neumann or parallel computer.

There is a well-developed theory of the effects of inputs on the internal states of a finite-state automaton: the Krohn–Rhodes theory. [For an exposition, cf. Kalman et al. 1969]. Concatenated inputs act as a semigroup of transformations on the state space of the automaton. The semigroup can be decomposed, a la Krohn–Rhodes, into simple groups and special semigroups that represent the state transformations of a flip-flop. The latter is known to be an intrinsically dissipative circuit (Landauer, 1961; Bennett, 1973). Thus a discussion of the physics of finite-state automata points toward thermodynamics rather than quantum mechanics.

Our quantum bound of information transfer affects internal transmission between subunits (if any), the rate at which information may enter and the rate at which output can be observed. In contrast, dissipation (entropy production) is affected by the structure of the automaton itself. If an automaton contains flip-flops it will generate entropy when the flip-flops are activated. However, dissipative circuitry can, in principle, be replaced by nondissipative circuitry that is capable of the same input–output transformation as the original circuitry (Fredkin and Toffoli 1981; Bennett, 1973). The nondissipative circuitry, however, would have a different state space. This author proposed (Bremermann, 1974a) that a finite-state automaton whose state space transformation semigroup is actually a group need not necessarily generate entropy, while an automaton whose associated semigroup contains noninvertible transformations must dissipate entropy for some inputs. (A unification of automata theory with network thermodynamics seems called for. Cf. Oster et al. 1973, Perelson, 1975).

To accomodate analog computation in a general sense (not just differential analyzers) we propose to utilize the concept of a dynamical system with controls (cf. Kalman et al. 1973; Bremermann, 1974b). In a nutshell, this concept generalizes the concept of finite-state automaton to automata with potentially infinite state spaces, and continuous input and outputs. In view of our quantum limit the maximum rate of information input and output is still finite and bounded even though a continuum of input and output functions are possible. In other words, the quantum limit applies to the input and output and to internal transmission of information between subunits.

In other words, the minimum energy requirement for transmission affects internal transmission between subunits of computers of any kind: Von Neumann-type, parallel, digital, and computers that process continuous signals (analog computers). Computers may or may not be dissipative.

The minimum energy requirement for transmission says nothing about dissipation and energy could be reused. When dissipation is studied additional constraints may arise. Thus the minimum energy requirement is more akin to the first law of thermodynamics than to the second law.

7. THE PERCEPTION OF COMPLEXITY

The author's concern about complexity and ultimate limits of computing was stimulated by seminars in artificial intelligence, beginning in 1959. The now flourishing mathematical complexity theory was virtually nonexistent at that time. However, it soon became apparent that proving theorems and playing games (such as chess or go) led to the search of trees (state space representations) which always seemed to grow exponentially. Exponential growth quickly exceeds available computing power, even as technology advances. At that time mathematical logic paid virtually no attention to complexity questions (limiting concern to a distinction between *finite* and, ironically, different degrees of infinity).

Stimulated by Brillouin (1956), Landauer (1961), and Von Neumann (1958) it soon became clear to the author that physics might impose ultimate limits on computing that would affect fundamentally human ability to penetrate irreducibly complex problems. This realization generates a multiple challenge: (1) to understand the limits of computation, (2) to determine which problems are really irreducibly complex and which can be simplified, and (3) to determine the implications for epistemology, such as the predictability of complex physical and biological systems.

After a slow start in the sixties, mathematical complexity theory has flourished in the seventies. As one may expect, there really are irreducibly complex problems (cf. Stockmeyer and Chandra, 1979). The impetus for much of this work seems to have come from computer science and operations research rather than from the traditional mathematical community (cf. Knuth, 1976).

Landauer, (1961, 1973, 1976), Bennett (1973), Fredkin and Toffoli (1981), Ashby (1967, 1968, 1973), and the author (1962a, b, 1967a, 1977) are among the people who maintained an interest in the physics of computation since 1961.

Ashby emphasized the epistemological consequences of the limit for systems theory and cybernetics in several subsequent papers (cf. Ashby, 1967, 1973).

The author was well aware of the preliminary nature of his result and he attempted to improve his argument in (1967a); however he found the physics literature on the time–energy uncertainty relation difficult to interpret. As the papers by Allcock (1969) and Wigner (1972) show, these difficulties are rather fundamental. Allcock even expressed pessimism about the consistency of the time–energy uncertainty principle with quantum mechanics.

In 1967 the author became aware of the work of L. Levitin (1965) on the transmission of information over photon channels. His work initially seemed to disagree with the author's, but seems based on somewhat different assumptions. For a bibliography of his work see his paper in these proceedings (Levitin, 1982).

Various other authors (Ligomenides, 1967, Keyes, 1982) have pursued similar ideas, though in less generality and similarly affected by difficulties with a precise interpretation of the time–energy uncertainty principle. To the author's surprise his 1962 limit has how been derived from results of Bekenstein, Gibbons, and Hawking on the entropy of black holes. See Bekenstein 1980 for a history of the idea that black holes have entropy.

Hawking's work successfully combines general relativity with quantum effects. Bekenstein's work on the entropy of black holes could not proceed without this breakthrough. Perhaps the meaning of it all is that the second law of thermodynamics really is a quantum phenomenon and that the time–energy uncertainty principle remains incomplete unless it is embedded into general relativity.

This author has mentioned cosmological aspects (e.g., less than 10^{121} bits could have been transmitted in the universe since the big bang Bremermann, 1977; cf. also Bremermann, 1962), but this aspect had been secondary. In the work of Bekenstein the order of inference is reversed.

We note that complexity phenomena also play a fundamental role in genetics and evolution (Bremermann, 1963; Maynard Smith, 1979; Lumsden and Wilson, 1981).

8. GEOMETRIZATION OF MOMENTUM SPACE?

Perhaps general relativity is likewise relevant to a better understanding of the difficulties that arise from divergent Feynman integrals, difficulties which are overcome by renormalization procedures which go back to the pioneering work of Dyson. The author worked with divergent Feynman integrals around 1960 (Bremermann, 1959) and related them to the multiplication of Schwartz distributions (generalized functions) (Bremermann, 1963). Outside physics such divergencies can often be blamed on excessive idealization (cf. Bremermann, 1967b). Distribution (δ-function-type) singularities and divergences arise only when integration in Fourier space is extended to infinity.

In relativistic quantum field theory momentum space expansions in a flat Minkowski space are very convenient. Integration to infinity, however, heads to divergencies or undefined products of distributions. Renormalization procedures take care of these difficulties. However, if momentum space were finite, like position space in general relativity, then infinities would disappear. Such "cut-off physics" was unpopular around 1960. In view of the fact that in a bounded universe mass, energies, and momenta are likewise bounded it would not seem unreasonable that attention to these bounds could be fruitful. What are the analogies of "black holes" in momentum space? Why has position space been geometrized while momentum space has remained with the geometry of the flat Minkowski space of special relativity? Fourier transforms correspond to the character group of position space. Only for a flat position space is the character group also a flat space. Is the operator–commutator formalism exempt from critical reexamination? Is it reasonable to reexamine to concept of *observer*? In the author's theory, as set forth in this paper, computer subunits, receiving and interpreting data, are on an equal footing with human observers. Perhaps it would be worthwhile to pursue this subject further.

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REFERENCES

Allcock, G. R. (1969). "The time of arrival in quantum mechanics, I. Formal considerations, *Annals of Physics*, **53**, 253–285; "II. The individual measurement," *Annals of Physics*, **53**, pp. 286–310; III. "The measurement ensemble," *Annals of Physics*, **53**, pp. 311–348.

Arbib, M. A. (1964). *Brains, Machines, and Mathematics*. McGraw-Hill, New York.

Ashby, R. (1967). "The place of the brain in the natural world," *Currents in Modern Biology*, **1**, 95–104.

Ashby, R. (1968). "Some consequences of Bremermann's limit for information processing systems," in *Cybernetic Problems in Bionics* (Bionics Symposium, 1966), H. L. Oestreicher and D. R. Moore, eds. Gordon & Breach, New York.

Ashby, R. (1973). Editorial, *Behavioral Science*, **18**, 2–6.

Bekenstein, J. D. (1973), "Black holes and entropy," *Physical Review D*, **7**, 2333.

Bekenstein, J. D. (1980). "Black-hole thermodynamics," *Physics Today*, Jan., 24–31.

Bekenstein, J. D. (1981a). "The energy cost of information transfer," *Physical Review Letters*, **46**, 623.

Bekenstein, J. D. (1981b) Universal Upper Bound to Entropy-to-Energy Ratio for Bounded Systems. *Physical Review D*, **23**, 287.

Bennett, C. H. (1973). "Logical reversibility of computation," *IBM Journal of Research and Development*, **17**, 525–532.

Berestetskii, V. B., Lifshitz, E. M., and Pitaevskii, L. P. (1971). *Relativistic Quantum Theory*, Vol. 4, Course of Theoretical Physics, Part I. Pergamon Press, Oxford.

Bledsoe, W. W. (1961). "A basic limitation of the speed of digital computers," *IRE Transactions Electronic Computers*, **EC-10**, 530.

Bremermann, H. J. (1959). "On finite renormalization constants and the multiplication of causal functions in perturbation theory," Technical Report, Department of Mathematics, University of California, Berkeley, California.

Bremermann, H. J. (1962a). "Part I: Limitations on data processing arising from quantum theory," in Optimization through evolution and recombination," in *Self-organizing Systems*, M. C. Yovits, G. T. Jacobi, and G. D. Goldstein, eds. Spartan Books, Washington, D.C.

Bremermann, H. J. (1962b). Quantum-theoretical Limitations of Data Processing, *Abstracts of Short Communications*, International Congress of Mathematics, Stockholm.

Bremermann, H. J. (1963). "Limits of genetic control," *IEEE Transactions on Military Electronics*, **7**, 200.

Bremermann, H. J. (1965). *Distributions, Complex Variables, and Fourier Transforms*. Addison-Wesley, Reading, Massachusetts.

Bremermann, H. J. (1967a). "Quantum Noise and Information," in *Proceedings of the Fifth Berkeley Symposium on Mathematical Statistics and Problability*, University of California Press, Berkeley, California.

Bremermann, H. J. (1967b). "Some remarks on analytic representation and products of distributions, *SIAM Journal Applied Mathematics*, **15**, 929.

Bremermann, H. J. (1974a). "Algorithms, complexity, transcomputability, and the analysis of systems," in *Cybernetics and Bionics*, W. D. Keidel, W. Handler, and M. Spreng, eds. Oldenbourg, Munich.

Bremermann, H. J. (1974b). "Complexity of automata, brains and behavior," in *Physics and Mathematics of the Nervous System*, M. Conrad, W. Güttinger, and M. Dal Cin, eds., *Biomathematics Lecture Notes*, Vol. 4. Springer, Heidelberg.

Bremermann, H. J. (1977). "Complexity and transcomputability," in *The Encyclopaedia of Ignorance*, M. Duncan, ed. Pergamon Press, Oxford.

Brillouin, L. (1956). *Science and Information Theory*. Academic Press, London; second ed., 1962.

Csanky, L. (1976). "Fast parallel matrix inversion algorithms," *SIAM Journal on Computing*, **5**.

Csanky, L. and Bremermann, H. J. (1976). Complexity of Parallel Computation, Arbeitsberichte IMMD, University of Erlangen—Nürnberg, Vol. **9**, No. 8, 31.

Feynman, R. P., Leighton, R. B., and Sands, M. (1963). *The Feynman Lectures on Physics*. Addison-Wesley, Reading, Massachusetts.

Fredkin, E., and Toffoli, T. (1981). Conservative logic, Technical Memo, MIT/LCS/TM-197 (April 29).

Hawking, S. W. (1974). "Black hole explosions?" *Nature*, **248**, 30.

Hawking, S. W. (1975). "Particle creation by black holes," *Communications in Mathematical Physics*, **43**, 199.

Hawking, S. W. (1976). "Black holes and thermodynamics," *Physical Review D*, **13**, 191.

Kalman, R. E., Falb, P. L., and Arbib, M. A. (1969). *Topics in Mathematical Systems Theory*. McGraw-Hill, New York.

Keyes, R. W. (1982). *International Journal of Theoretical Physics*, **21**, 263 (this issue).

Knuth, D. E. (1976). "Mathematics and computer science: Coping with finiteness," *Science*, **194**, 1235–1242.

Landauer, R. (1961). "Irreversibility and heat generation in the computing process," *IBM Journal of Research and Development*, **5**, 183–191.

Landauer, R. (1976). "Fundamental limitations in the computational process," *Berichte der Bunsengesellschaft für Physikalische Chemie*, **80**, 1048–1059.

Landauer, R., and Woo, J. W. F. (1973). "Cooperative phenomena in data processing," in *Synergetics*, H. Haken, ed. B. G. Teubner, Stuttgart.

Levitin, L. B. (1965). "Transmission of information by an ideal photon channel," *Information Transmission Problems*, **1**, 71; "Ideal physical transmission channel," *Information Transmission Problems*, **1**, 122.

Levitin, L. B. (1982). *International Journal of Theoretical Physics*, **21**, 299 (this issue).

Ligomenides, P. L. (1967). "Wave-mechanical uncertainty and speed limitations," *IEEE Spectrum*, **4**:2, 65–68.

Lumsden, Ch. J., and Wilson, E. O. (1981). *Genes, Mind, and Culture: The Coevolutionary Process*, Harvard University Press, Cambridge, Massachusetts.

Maynard Smith, J. (1979). "The limitations of evolution theory," in *Encyclopaedia of Ignorance*, R. Duncan and M. Weston-Smith, eds. Pergamon Press, Oxford.

Miller, R. E., and Thatcher, J. W. (eds.) (1972). *Complexity of Computer Computations*. Plenum Press, New York.

Oster, G., Perelson, A. S., and Katchalsky, A. (1973). "Network thermodynamics: Dynamic modelling of biophysical systems," *Quarterly Review of Biophysics*, **6**, 1.

Papoulis, A. (1962). *The Fourier Integral and its Applications*. McGraw-Hill, New York.

Perelson, A. S. (1975). "Network thermodynamics, an overview," *Biophysical Journal*, **15**, 667–685.

Shannon, C. E. (1948). "A mathematical theory of communication," *Bell System Technical Journal*, **27**, pp. 379–423, 623–656.

Stockmeyer, L. J., and Chandra, A. K. (1979). "Intrinsically difficult problems," *Scientific American*, **240**, 5 (May), 140.

Wigner, E. P. (1972), "On the time–energy uncertainty relation," in *Aspects of Quantum Theory*, A. Salam, and E. P. Wigner, eds. Cambridge University Press, Cambridge.